The scaling laws of transport properties for fractal-like tree networks

Peng Xu and Boming Yu[a)]
Department of Physics, Huazhong University of Science and Technology, 1037 Luoyu Road, Wuhan 430074, People's Republic of China

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The scaling laws of transport properties are very important for understanding the transport mechanisms of the fractal-like tree networks, which have received extensive attention recently. In this paper, we analyze the transport properties including electrical conductivity, heat conduction, convective heat transfer, laminar flow, and turbulent flow in the networks and also derive the scaling exponents of the transport properties in the networks. We show that the scaling laws are different for different transport processes and the scaling exponents are sensitive to microstructures of the networks. The models and results we present here may shed light on the transport mechanisms of the networks such as the natural systems, nanotube networks, microelectronic cooling networks, organisms, fracture networks in oil/water reservoirs, seepage flow in porous networks/media, etc., and might provide guidance for design of composites with the tree structures. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2392935]

I. INTRODUCTION

The tree network structures arise extensively in circulatory systems of plants and mammals, communications, social cooperation networks, economic systems, river basins, oil/water reservoirs, power supply systems of cities, microelectronic cooling systems, etc.[1,2] The transport properties (electrical, thermal, fluid flow, etc.) in the tree networks, the design, optimization, and application of the tree networks have received considerable attention recently.[3–19] Since Murray's law[3] for cardiovascular system in 1926, various and numerous subsequent researches have focused on the investigation of the networks. Many researchers use this tree network model to explain the mechanisms of natural transports.[7–9,12,15] Some others design and optimize the networks and apply the tree networks to various engineering fields such as oil/water reservoirs, power supply systems of a city, microelectronic cooling systems, etc. For example, Bejan and co-workers[4–6] developed "constructal theory" by optimizing the access between one point and a finite volume. Cheng and Chen[11] designed a rectangular shape fractal-tree nets fixed on the top and bottom of matrix wafer for the cooling of a microelectronic chip. Yu and Li[19] investigated the effective thermal conductivity of composites with embedded self-similar H-shaped fractal-like tree networks. They found that the fractal-like tree networks play an important role and can significantly reduce the thermal conductivity. This structure might be applied for designing insulating or conducting structures such as space equipments.

The self-similar fractal-tree network can be space filling,[20] which has been shown significantly more error (variability) tolerant than other structures and therefore has an evolutionary advantage.[21] The scaling laws of transport properties of the fractal-like tree network are very important and may shed light on the natural network structures. Neuman[22] showed that the hydraulic conductivity K' in porous media scales with the support volume V by $K' \sim V^a$, where a is the scaling exponent and takes on specific value $1/2$. Winter and Tartakovsky[23] demonstrated that when a pore network in porous media can be represented by a collection of hierarchical trees, scalability of the pore geometry leads to the $1/2$-power scaling law. While West *et al.*[7–9] derived the allometric scaling law which is a quarter-power scaling law. Allometric laws are widely recognized as power-law relations between geometric and functional (e.g., flow) parameters of living bodies. From a purely theoretical standpoint, Bejan[24] found that the total heat current convected by a double tree is proportional to the total volume raised to power $3/4$. In this paper, we focus on the transport properties of the fractal-like tree networks and study their scaling laws, including those of electrical conductivity, heat conduction, convective heat transfer, laminar flow, and turbulent flow. These might be useful for understanding the transport mechanisms of the tree structures such as the natural systems, nanotube networks, microelectronic cooling networks, organisms, fracture networks in oil/water reservoirs, seepage flow in porous networks/media, etc.

II. FRACTAL-LIKE TREE NETWORK

Figure 1 displays a typical self-similar tree structure and the kth branching level with n (=2) branches. Since the present work deals with the self-similar tree networks with finite volume branches, there is a possible smallest element/unit because the daughter branches may touch mother stems after finite repeats. Therefore, the structures in this paper are called fractal-like tree networks. If, however, the volume of all branches is neglected or stems are assumed to be infinitely thin, the self-similar tree structures are fractals.[20]

Generally speaking, the tree/branched network (see Fig. 1) is composed of N branches from level 0 to levels m (total number of branching levels). A typical branch with branching angle θ at some intermediate level k ($k=0,1,2,\ldots,m$)

[a)]Author to whom correspondence should be addressed; electronic mail: yuboming2003@yahoo.com.cn

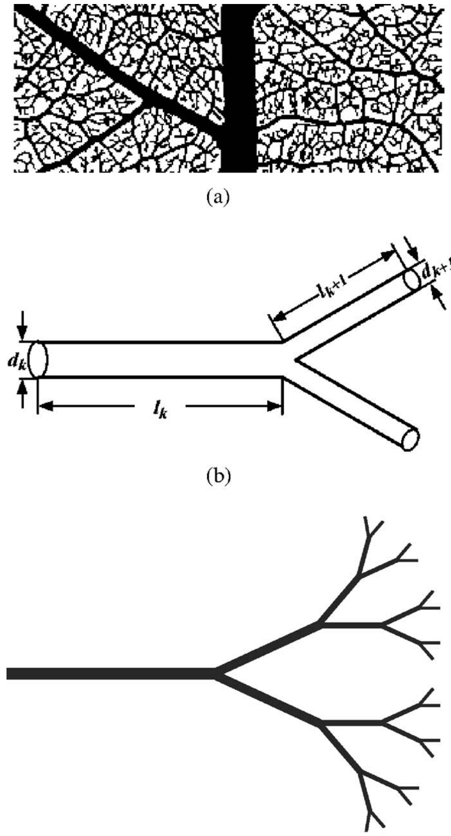

FIG. 1. (a) A typical self-similar tree structure (Ref. 10), (b) the kth branching level with $n(=2)$ branches, and (c) a fractal-like tree network.

has length l_k and diameter d_k, and each tube branches into n_k smaller daughter branches at the next level. In order to characterize the architecture of the tree/branched network, we introduce scale factors $\gamma_k \equiv l_{k+1}/l_k$ and $\beta_k \equiv d_{k+1}/d_k$.

In this paper, we focus on the tree/branched network with fractal characteristics; thus we shall invoke the self-similar fractal results by West *et al.*,[7] i.e., $\gamma_k=\gamma$, $\beta_k=\beta$, and $n_k=n$ which are all independent of k. It is easy to get

$$l_k = l_0\gamma^k = l_m\gamma^{k-m} \quad \text{and} \quad d_k = d_0\beta^k = d_m\beta^{k-m}, \tag{1}$$

where l_0 and d_0 are the length and diameter of the 0th branching level while l_m and d_m are the length and diameter of the terminal branching level. According to the fractal characteristics of the structure,[20] we have

$$n = \gamma^{-D_l} = \beta^{-D_d}, \tag{2}$$

where D_l is the fractal dimension of tube length distribution, and D_d is the fractal dimension for diameter distribution (also called diameter exponent). Fractal dimensions D_l and D_d are generally limited between 1 and 3, and the value $D_d=3$ is an optimized result known as Murray's law. Then, for $n=2$, γ and β are in the ranges of 0.5–0.7937.

The total number of branches at level k is $N_k=n^k$ and the total number of branches of the whole network is $N=\sum_{k=0}^{m}N_k=(1-n^{m+1})/(1-n)$. For convenience, we assume that every branching tube is smooth cylinder. Therefore, the total volume of the network can be obtained as

$$V = \sum_{k=0}^{m} N_k V_k = \sum_{k=0}^{m} n^k \pi (d_k/2)^2 l_k = V_m n^m \frac{1-(n\gamma\beta^2)^{-(m+1)}}{1-(n\gamma\beta^2)^{-1}}, \tag{3}$$

where V_m is the volume of a terminal branching tube. Consequently, the dimensionless volume can be obtained by

$$V^+ = \frac{V}{V_m} = n^m \frac{1-(n\gamma\beta^2)^{-(m+1)}}{1-(n\gamma\beta^2)^{-1}}. \tag{4}$$

In the following discussion, we follow the assumption by West *et al.*[7] that the final branch of the network (such as the capillary in the circulatory system) is a size-invariant unit. That is, the terminal units (capillaries) are invariant and the parameters (l_m, d_m, v_m, Δp_m, R_m, etc.) of the terminal branch are independent of body size. The fractal branching network presented here should be viewed as an idealized representation, in which we ignore complications such as tapering of vessels and nonlinear effects. These play only a minor role in determining the properties of the entire network and could be incorporated in more detailed analysis of specific systems. For convenience, we only consider the transports along the tubes of the network and neglect the transports in the matrix material.

III. ELECTRICAL CONDUCTIVITY

In this section, we will derive the scaling law between the electrical conductivity and the volume of the network. The electrical resistance of a kth single tube is $R_k=\rho l_k/A_k=4\rho l_k/\pi d_k^2$, where ρ is the resistivity of a material. The electrical resistance of the entire network is given by

$$R = \sum_{k=0}^{m} \frac{R_k}{N_k} = \sum_{k=0}^{m} \frac{4\rho l_k}{\pi n^k d_k^2} = \frac{R_m}{n^m} \frac{1-(n\beta^2/\gamma)^{m+1}}{1-n\beta^2/\gamma}. \tag{5}$$

The effective resistivity of the network can be obtained from $\rho_e=RA_e/l_e$, where the equivalent length $l_e=\sum_{k=0}^{m}l_k=l_m[1-\gamma^{-(m+1)}]/(1-\gamma^{-1})$, and the equivalent cross-sectional area of the network is defined as $A_e=V/l_e=(\pi d_m^2/4)n^m(1-\gamma^{-1})/[1-\gamma^{-(m+1)}][1-(n\gamma\beta^2)^{-(m+1)}]/[1-(n\gamma\beta^2)^{-1}]$. Then the effective resistivity of the network can be expressed as

$$\rho_e = \rho\left[\frac{1-\gamma^{-1}}{1-\gamma^{-(m+1)}}\right]^2 \frac{1-(n\gamma\beta^2)^{-(m+1)}}{1-(n\gamma\beta^2)^{-1}} \frac{1-(n\beta^2/\gamma)^{m+1}}{1-n\beta^2/\gamma}. \tag{6}$$

The effective electrical conductivity (EEC) of the network can be obtained as

$$\sigma_e = \frac{1}{\rho_e} = \frac{1}{\rho}\left[\frac{1-\gamma^{-(m+1)}}{1-\gamma^{-1}}\right]^2 \frac{1-(n\gamma\beta^2)^{-1}}{1-(n\gamma\beta^2)^{-(m+1)}} \frac{1-n\beta^2/\gamma}{1-(n\beta^2/\gamma)^{m+1}}. \tag{7}$$

The dimensionless EEC of the network is

$$\sigma^+ = \frac{\sigma_e}{\sigma} = \left[\frac{1-\gamma^{-(m+1)}}{1-\gamma^{-1}}\right]^2 \frac{1-(n\gamma\beta^2)^{-1}}{1-(n\gamma\beta^2)^{-(m+1)}} \frac{1-n\beta^2/\gamma}{1-(n\beta^2/\gamma)^{m+1}}, \quad (8)$$

where $\sigma=1/\rho$ is the electrical conductivity of material of the network. It can be clearly seen from Eqs. (7) and (8) that the EEC of the network is equal to the material electrical conductivity of the network at $\beta=n^{-1/2}$ ($D_d=2$, area-preserving constraint), i.e., $\sigma_e=\sigma$.

Due to Eq. (3) and $n\gamma\beta^2<1$, for $m\gg 1$ the total volume V is approximately proportional to $(\gamma\beta^2)^{-(m+1)}$. As $n\beta^2/\gamma>1$, the total electrical resistance is approximately proportional to $(\beta^2/\gamma)^{m+1}$, and the EEC is proportional to a constant. From Eqs. (3) and (5), we get

$$\frac{\ln R}{\ln V} = -\frac{\ln(\beta^2/\gamma)}{\ln(\gamma\beta^2)}, \quad (9)$$

which means that the total electrical resistance R of the fractal branched network is proportional to V^a, where the scaling exponent $a=-\ln(\beta^2/\gamma)/\ln(\gamma\beta^2)$. According to Eq. (2), the scaling exponent can also be expressed as $a=(D_d-2D_l)/(D_d+2D_l)$. The employment of the volume-preserving (space-filling fractal system $\gamma=n^{-1/3}$) and area-preserving ($\beta=n^{-1/2}$) conditions[7–9] from one generation to the next, which correspond to fractal dimensions $D_l=3$ and $D_d=2$, respectively, results in $a=-1/2$. Thus, the scaling law $R\sim V^{-1/2}$ holds, which means that the $-1/2$-power scaling law exists between the electrical resistance and the volume under the volume- and area-preserving conditions.

However, when $n\beta^2/\gamma<1$, the total electrical resistance is approximately proportional to $(1/n)^m$ and the effective electrical conductivity is approximately proportional to $(n\beta^2/\gamma)^{m+1}$. Therefore,

$$\frac{\ln R}{\ln V} = \frac{-m\ln n}{-(m+1)\ln(\gamma\beta^2)} \cong \frac{\ln n}{\ln(\gamma\beta^2)} \quad \text{and} \quad \frac{\ln \sigma_e}{\ln V} = -\frac{\ln(n\beta^2/\gamma)}{\ln(\gamma\beta^2)}. \quad (10)$$

Equation (10) indicates that the electrical resistance and the EEC are proportional to V^a and V^b, respectively, where the scaling exponents $a=\ln n/\ln(\gamma\beta^2)$ and $b=-\ln(n\beta^2/\gamma)/\ln(\gamma\beta^2)$. And the scaling exponents can also be written as $a=-D_lD_d/(D_d+2D_l)$ and $b=(D_d-2D_l+D_lD_d)/(D_d+2D_l)$.

IV. HEAT CONDUCTION

With the increasing miniaturization of electronic equipments, a lot of investigations were involved in heat transfer in the tree networks.[11,19,24,25] Xu *et al.*[25] analyzed the heat conduction through "Y" shaped fractal-like tree networks and obtained the expression for thermal conductivity in the networks. The network is assumed to be composed of the material of high thermal conductivity λ, which is much higher than that of the material around the channels. So, heat conduction along the channels is assumed and in other directions is neglected. This means that the one dimensional heat flow model is applied. The total thermal resistance of the entire network can be obtained by the thermal-electrical analogy as[25]

$$R = \sum_{k=0}^{m} \frac{R_k}{N_k} = \frac{R_m}{n^m} \frac{1-(n\beta^2/\gamma)^{m+1}}{1-n\beta^2/\gamma}. \quad (11)$$

And then the effective thermal conductivity (ETC) is obtained by equating that of the network with an equivalent single tube as

$$\lambda_e = \lambda\left[\frac{1-\gamma^{-(m+1)}}{1-\gamma^{-1}}\right]^2 \frac{1-(n\gamma\beta^2)^{-1}}{1-(n\gamma\beta^2)^{-(m+1)}} \frac{1-n\beta^2/\gamma}{1-(n\beta^2/\gamma)^{m+1}}. \quad (12)$$

The dimensionless ETC is written as

$$\lambda^+ = \frac{\lambda_e}{\lambda} = \left[\frac{1-\gamma^{-(m+1)}}{1-\gamma^{-1}}\right]^2 \frac{1-(n\gamma\beta^2)^{-1}}{1-(n\gamma\beta^2)^{-(m+1)}} \frac{1-n\beta^2/\gamma}{1-(n\beta^2/\gamma)^{m+1}}. \quad (13)$$

Equation (13) shows that the ETC reaches its greatest value, which is equal to the thermal conductivity of the tube material at the area-preserving condition, i.e., $\beta=n^{-1/2}$ ($D_d=2$). Compared with Eqs. (8) and (13), it is seen that the ETC and the EEC of the network have the same form, so the scaling law of the heat conduction is the same as that of electrical conductivity, and this is expected.

V. CONVECTIVE HEAT TRANSFER

The effective convective heat transfer coefficient of the whole fractal-like tree network is derived in this section. The flow through each tube is assumed to be laminar and fully developed both thermally and hydrodynamically, and the Nusselt number remains constant through each level of branches. Thus, the convective heat transfer coefficient of the higher-level branching will increase with $h_{k+1}/h_k=d_k/d_{k+1}=1/\beta$; consequently,

$$h_k = h_m\beta^{m-k}. \quad (14)$$

The heat flux through the network is assumed to be uniform and varies from one level to another. Therefore, a fully developed flow with constant heat flux in uniform cross section yields a constant temperature different ΔT between the wall surface and bulk flow. Consequently, the total convective heat transfer rate is

$$Q = \sum_{k=0}^{m} N_k h_k S_k \Delta T = n^m Q_m \frac{1-(n\gamma)^{-(m+1)}}{1-(n\gamma)^{-1}}, \quad (15)$$

where $S_k=\pi d_k l_k$ is the heat transfer area at the kth branching level and $Q_m=h_m\pi d_m l_m\Delta T$ is the convective heat transfer rate at the mth branching level. The total heat transfer area S of the network is given by

$$S=\sum_{k=0}^{m} N_k S_k=\sum_{k=0}^{m} n^k \pi d_k l_k = n^m S_m \frac{1-(n\gamma\beta)^{-(m+1)}}{1-(n\gamma\beta)^{-1}}. \quad (16)$$

Then the effective convective heat transfer coefficient of the whole network can be calculated from $h_e=Q/S\Delta T$, i.e.,

$$h_e=h_m\frac{1-(n\gamma)^{-(m+1)}}{1-(n\gamma)^{-1}}\frac{1-(n\gamma\beta)^{-1}}{1-(n\gamma\beta)^{-(m+1)}}. \quad (17)$$

The dimensionless of h_e can be easily found to be

$$h^+=\frac{h_e}{h_m}=\frac{1-(n\gamma)^{-(m+1)}}{1-(n\gamma)^{-1}}\frac{1-(n\gamma\beta)^{-1}}{1-(n\gamma\beta)^{-(m+1)}}. \quad (18)$$

Generally, $n\gamma>1$ and $m\gg 1$, then as $n\gamma\beta>1$, h_e is approximately proportional to a constant. While as $n\gamma\beta<1$, h_e and S are approximately proportional to $(n\gamma\beta)^{m+1}$ and $(\gamma\beta)^{-(m+1)}$, respectively. Thus h_e is proportional to $S^{b'}$, where the scaling exponent $b'=-\ln(n\gamma\beta)/\ln(\gamma\beta)$. Because the total volume V is approximately proportional to $(\gamma\beta^2)^{-(m+1)}$ for $n\gamma\beta^2<1$ [see Eq. (3)], h_e is proportional to V^b, where the scaling exponent $b=-\ln(n\gamma\beta)/\ln(\gamma\beta^2)$. Due to Eq. (2) the scaling exponents can also be expressed in terms of the fractal dimensions as $b'=D_lD_d/(D_l+D_d)-1$ and $b=(D_lD_d-D_d-D_l)/(D_d+2D_l)$.

VI. FLUID FLOW

A. Laminar flow

Let us first consider the case of incompressible fully developed laminar flow. The viscous resistance for flow in a single tube is given by the Poiseuille formula $R_k=128\,\mu l_k/(\pi d_k^4)$, where μ is the viscosity of fluid. Then the total flow resistance of the network can be expressed as

$$R=\sum_{k=0}^{m}\frac{R_k}{N_k}=\frac{R_m}{n^m}\frac{1-(n\beta^4/\gamma)^{m+1}}{1-n\beta^4/\gamma}. \quad (19)$$

Then, using the method applied in literatures,[26,27] we can derive the effective permeability of the network

$$K_e=K_m\left[n^m\frac{1-(1/\gamma)^{m+1}}{1-1/\gamma}\frac{1-n\beta^4/\gamma}{1-(n\beta^4/\gamma)^{m+1}}\right]^{1/2}\frac{1}{T}, \quad (20)$$

where K_m is the permeability of the mth branching level and T is the tortuosity of the network (for example, $T=(1-\gamma^{m+1})/\{(1-\gamma)[1+\gamma(1-\gamma^m)\cos\theta/(1-\gamma)]\}$ for the fractal-like tree network between one point and a straight line).[26] The dimensionless form of the effective permeability can then be expressed as

$$K^+=\frac{K_e}{K_m}=\left[n^m\frac{1-(1/\gamma)^{m+1}}{1-1/\gamma}\frac{1-n\beta^4/\gamma}{1-(n\beta^4/\gamma)^{m+1}}\right]^{1/2}\frac{1}{T}. \quad (21)$$

As $m\gg 1$ and $n\beta^4/\gamma<1$, a good approximation to Eq. (19) is $R=R_m/(1-n\beta^4/\gamma)n^m$, and because R_m is invariant, $R\sim n^{-(m+1)}$. But as $n\beta^4/\gamma>1$, $R\sim(\beta^4/\gamma)^{m+1}$. The total volume $V\sim(\gamma\beta^2)^{-(m+1)}$ for $n\gamma\beta^2<1$. So, the total flow resistance of the network R is proportional to V^a, where the scaling exponents $a=\ln n/\ln(\gamma\beta^2)$ as $n\beta^4/\gamma<1$ and $a=-\ln(\beta^4/\gamma)/\ln(\gamma\beta^2)$ as $n\beta^4/\gamma>1$. The scaling exponents can also be expressed as the functions of fractal dimensions, $a=-D_lD_d/(D_d+2D_l)$ and $a=(D_d-4D_l)/(D_d+2D_l)$. West *et al.*[7–9] obtained the famous allometric scaling law by assuming $n\beta^4/\gamma<1$ and gave the power law $R\sim V^{-3/4}$ at the volume- and area-preserving limitations. Our result is consistent with the allometric scaling law as $n\beta^4/\gamma<1$.

It has been shown[26,27] that when $m\gg 1$, the tortuosity T approaches different asymptotic values for different branching angles θ. This implies that the tortuosity and the branching angle θ have no effect on the scaling law of permeability as $m\gg 1$. As $n\beta^4/\gamma<1$ and $m\gg 1$, the effective permeability of the network K_e is proportional to $(n/\gamma)^{(m+1)/2}$, and as $n\beta^4/\gamma>1$ and $m\gg 1$, K_e is proportional to $\beta^{-2(m+1)}$ [see Eq. (20)]; while the total volume V is proportional to $(\gamma\beta^2)^{-(m+1)}$ for $n\gamma\beta^2<1$ [see Eq. (3)]. Therefore, the scaling law of permeability with the support volume can be written as

$$K_e\sim V^b, \quad (22)$$

where the scaling exponents $b=-(1/2)\ln(n/\gamma)/\ln(\gamma\beta^2)$ as $n\beta^4/\gamma<1$ and $b=2\ln\beta/\ln(\gamma\beta^2)$ as $n\beta^4/\gamma>1$. The scaling exponents with fractal dimensions are $b=(1/2)D_d(1+D_l)/(D_d+2D_l)$ and $b=2D_l/(D_d+2D_l)$. It can be clearly seen that the scaling exponent b is independent of the branching angle θ, i.e., although the permeability is strongly related to branching angle θ, the scaling laws of the permeability are independent of the branching angle θ when $m\gg 1$.

According to the area-preserving relation the scaling exponent $b=1/2$, i.e., $K_e\sim V^{1/2}$, which is the same as that by Neuman[22] and Winter and Tartakovsky.[23] This means that the 1/2-power scaling law exists between the permeability and the volume under area-preserving condition. It is worth pointing out that our simple model yields the 1/2-power scaling law *only* under the area-preserving condition, whereas the same conductivity scaling arises under both the pore area and the pore length preserving conditions from one level to another in Winter and Tartakovsky's complicated model with subtrees.[23]

B. Turbulent flow

For the fully rough turbulent flow, the pressure drop over a kth branching tube of mean velocity v_k yields

$$\Delta P_k=f\frac{l_k}{d_k}\frac{\rho v_k^2}{2}, \quad (23)$$

where ρ is the density of fluid and the Darcy friction factor f is essentially constant in the fully rough turbulent limit, which is independent of Reynolds number or flow rate. Since the pressure drop ΔP_k is proportional to the flow rate $\dot{m}^2$, the flow resistance[28] is

$$R_k=\frac{\Delta P_k}{\dot{m}^2}=\frac{8f}{\rho\pi^2}\frac{l_k}{d_k^5}, \quad (24)$$

where $\dot{m}=\rho v_k A_k$. The total flow resistance of the network can be written as

$$R=\sum_{k=0}^{m}\frac{R_k}{N_k^2}=\frac{R_m}{n^{2m}}\frac{1-(n^2\beta^5/\gamma)^{m+1}}{1-n^2\beta^5/\gamma}. \quad (25)$$

From the conservation of mass, we have

$$Q = n^m v_m \pi \frac{d_m^2}{4} = n^k v_k \pi \frac{d_k^2}{4}, \tag{26}$$

and therefore

$$v_k = v_m n^{m-k} \beta^{2(m-k)}. \tag{27}$$

According to Eq. (23) and with the aid of Eq. (27), the total pressure drop of the network can be expressed as

$$\Delta P = \sum_{k=0}^{m} \Delta P_k = \frac{\rho f l_m v_m^2}{2 d_m} \frac{1-(n^2\beta^5/\gamma)^{m+1}}{1-n^2\beta^5/\gamma} \tag{28a}$$

$$= \Delta P_m \frac{1-(n^2\beta^5/\gamma)^{m+1}}{1-n^2\beta^5/\gamma}. \tag{28b}$$

The mean flow velocity of the network can be derived from Eq. (27),

$$v = \sum_{k=0}^{m} N_k v_k l_k \bigg/ \sum_{k-0}^{m} N_k l_k$$
$$= v_m \frac{1-(\beta^2/\gamma)^{m+1}}{1-\beta^2/\gamma} \frac{1-(n\gamma)^{-1}}{1-(n\gamma)^{-(m+1)}}. \tag{29}$$

The network can be equivalent to a single tube; thus,

$$\Delta P = f \frac{l_e}{d_e} \frac{\rho v^2}{2}, \tag{30}$$

where the equivalent length $l_e = l_m[1-\gamma^{-(m+1)}]/(1-\gamma^{-1})$, and d_e is the diameter of the equivalent single tube. Inserting Eqs. (28) and (29) into Eq. (30) gives

$$d_e = d_m \frac{1-(1/\gamma)^{m+1}}{1-1/\gamma} \frac{1-n^2\beta^5/\gamma}{1-(n^2\beta^5/\gamma)^{m+1}} \times \left[\frac{1-(\beta^2/\gamma)^{m+1}}{1-\beta^2/\gamma} \frac{1-(n\gamma)^{-1}}{1-(n\gamma)^{-(m+1)}}\right]^2. \tag{31}$$

Compared with the apparent Darcy law $Q = (K/\mu)A(dP/dL)$, the effective permeability of the network for fully rough turbulent flow can be obtained by

$$K_e = n^m K_m \frac{1-1/\gamma}{1-(1/\gamma)^{m+1}} \frac{1-(n^2\beta^5/\gamma)^{m+1}}{1-n^2\beta^5/\gamma} \times \left[\frac{1-\beta^2/\gamma}{1-(\beta^2/\gamma)^{m+1}} \frac{1-(n\gamma)^{-(m+1)}}{1-(n\gamma)^{-1}}\right]^4, \tag{32}$$

where K_e is the effective permeability of the network, and $K_m = 2\mu d_m/\rho f v_m$ is the permeability of the terminal single tube. In the above analysis, we have neglected the tortuousness of the network in derivation of the effective permeability. Usually the permeability is related to the tortuosity T.[29,30] However, as discussed above, since the tortuosity T has no influence on the scaling exponent as $m \gg 1$, T is assumed to be 1.0 for simplicity. The dimensionless effective permeability of the network can be easily obtained as

$$K^+ = \frac{K_e}{K_m} = n^m \frac{1-1/\gamma}{1-(1/\gamma)^{m+1}} \frac{1-(n^2\beta^5/\gamma)^{m+1}}{1-n^2\beta^5/\gamma} \times \left[\frac{1-\beta^2/\gamma}{1-(\beta^2/\gamma)^{m+1}} \frac{1-(n\gamma)^{-(m+1)}}{1-(n\gamma)^{-1}}\right]^4. \tag{33}$$

It can be clearly seen from Eqs. (32) and (33) that $K_e = K_m$ at $m=0$ or $n=1$ and $\beta=1$ (i.e., a single tube), and this is in accord with the practical situation.

As $m \gg 1$ and $n^2\beta^5/\gamma < 1$, a good approximation to Eq. (25) is that the total flow resistance R of the network is proportional to $n^{-2(m+1)}$. But, as $n^2\beta^5/\gamma > 1$, $R \sim (\beta^5/\gamma)^{m+1}$. While the total volume of the network is approximately proportional to $(\gamma\beta^2)^{-(m+1)}$ for $n\gamma\beta^2 < 1$. Therefore, R is proportional to V^a, where the scaling exponents $a = 2\ln n/\ln(\gamma\beta^2)$ or $a = -2D_l D_d/(D_d + 2D_l)$ as $n^2\beta^5/\gamma < 1$ and $a = -\ln(\beta^5/\gamma)/\ln(\gamma\beta^2)$ or $a = (D_d - 5D_l)/(D_d + 2D_l)$ as $n^2\beta^5/\gamma > 1$. Thus, the network exhibits the $-3/2$ power law between flow resistance and volume at the volume-preserving and area-preserving conditions for fully rough *turbulent* flow, and this is different from the $-3/4$ power law for *laminar* flow.

While for the effective permeability, since $\gamma < 1$, $\beta^2/\gamma < 1$, and $n\gamma > 1$, generally, the effective permeability of the network $K_e \sim (n\gamma)^{m+1}$ as $n^2\beta^5/\gamma < 1$. But as $n^2\beta^5/\gamma > 1$, $K_e \sim (n^3\beta^5)^{m+1}$. That means that the effective permeability of the network is proportional to V^b, where the scaling exponents $b = -\ln(n\gamma)/\ln(\gamma\beta^2)$ as $n^2\beta^5/\gamma < 1$ and $b = -\ln(n^3\beta^5)/\ln(\gamma\beta^2)$ as $n^2\beta^5/\gamma > 1$. Invoking Eq. (2) gives the exponents $b = D_d(D_l - 1)/(D_d + 2D_l)$ and $b = (3D_d - 5)/(D_d + 2D_l)$, respectively. As $\gamma = n^{-1/3}$ (space-filling fractal system) and $\beta = n^{-1/2}$ (area-preserving relation), $K_e \sim V^{1/2}$ for turbulent flow, which is the same as that for laminar flow only under the area-preserving condition ($\beta = n^{-1/2}$).

VII. RESULTS AND DISCUSSIONS

The scaling laws for different transport properties are summarized in Table I. The electrical conductivity and heat conduction in the fractal-like tree network have the same scaling law. That is, as $n\beta^2/\gamma > 1$, the EEC (or ETC) is proportional to a constant and the total resistance $R \sim V^a$, where the scaling exponent $a = -\ln(\beta^2/\gamma)/\ln(\gamma\beta^2)$. Under volume- and area-preserving relations, $R \sim V^{-1/2}$. When $n\beta^2/\gamma < 1$, the resistance and the EEC (or ETC) are proportional to V^a and V^b, respectively, where the scaling exponents $a = \ln n/\ln(\gamma\beta^2)$ and $b = -\ln(n\beta^2/\gamma)/\ln(\gamma\beta^2)$. For convective heat transfer, as $n\gamma\beta > 1$, h_e is proportional to a constant; as $n\gamma\beta < 1$, h_e is proportional to $S^{b'}$ and V^b, respectively, where the scaling exponents $b' = -\ln(n\gamma\beta)/\ln(\gamma\beta)$ and $b = -\ln(n\gamma\beta)/\ln(\gamma\beta^2)$.

For the laminar fluid flow, the resistance and effective permeability scale with the support volume by $R \sim V^a$ and $K_e \sim V^b$, respectively. In the case of laminar flow, the scaling exponent between resistance and the volume is $a = \ln n/\ln(\gamma\beta^2)$ as $n\beta^4/\gamma < 1$ and is $a = -\ln(\beta^4/\gamma)/\ln(\gamma\beta^2)$ as $n\beta^4/\gamma > 1$. The $-3/4$ power law exists between the flow resistance and the volume under the volume- and area-preserving relations. This is consistent with the famous allometric scaling law presented by West *et al.*[7] The scaling exponent between effective permeability and the volume is

TABLE I. Scaling laws for transport properties in the fractal branched networks. [The scaling exponents can also be expressed by the fractal dimensions of the network as discussed above via Eq. (2).]

Electrical and heat conductivity		Convective heat transfer		Laminar flow[a]		Turbulent flow	
$n\beta^2/\gamma>1$	$n\beta^2/\gamma<1$	$n\gamma\beta>1$	$n\gamma\beta<1$	$n\beta^4/\gamma<1$	$n\beta^4/\gamma>1$	$n^2\beta^5/\gamma<1$	$n^2\beta^5/\gamma>1$
$R\sim V^a$	$R\sim V^a$	$h_e\sim C$	$h_e\sim V^b$	$R\sim V^a$	$R\sim V^a$	$R\sim V^a$	$R\sim V^a$
$a=-\frac{\ln(\beta^2/\gamma)}{\ln(\gamma\beta^2)}$	$a=\frac{\ln n}{\ln(\gamma\beta^2)}$		$b=-\frac{\ln(n\gamma\beta)}{\ln(\gamma\beta^2)}$	$a=\frac{\ln n}{\ln(r\beta^2)}$	$a=-\frac{\ln(\beta^4/\gamma)}{\ln(\gamma\beta^2)}$	$a=\frac{2\ln n}{\ln(r\beta^2)}$	$a=-\frac{\ln(\beta^5/\gamma)}{\ln(\gamma\beta^2)}$
$\sigma_e\sim C$	$\sigma_e\sim V^b$		$h_e\sim S^{b'}$	$K_e\sim V^b$	$K_e\sim V^b$	$K_e\sim V^b$	$K_e\sim V^b$
	$b=-\frac{\ln(n\beta^2/\gamma)}{\ln(\gamma\beta^2)}$		$b'=-\frac{\ln(n\gamma\beta)}{\ln(\gamma\beta)}$	$b=-\frac{\ln(n/\gamma)}{2\ln(\gamma\beta^2)}$	$b=\frac{2\ln\beta}{\ln(\gamma\beta^2)}$	$b=-\frac{\ln(n\gamma)}{\ln(\gamma\beta^2)}$	$b=-\frac{\ln(n^3\beta^5)}{\ln(\gamma\beta^2)}$

[a]Our results $R\sim V^{-3/4}$ at $\gamma=n^{-1/3}$ and $\beta=n^{-1/2}$ is the same as the result of Ref. 7, $K_e\sim V^{1/2}$ at $\beta=n^{-1/2}$ is consistent with the conclusion of Refs. 22 and 23.

$b=-(1/2)\ln(n/\gamma)/\ln(\gamma\beta^2)$ as $n\beta^4/\gamma<1$ and is $b=2\ln\beta/\ln(\gamma\beta^2)$ as $n\beta^4/\gamma>1$. The relation $K_e\sim V^{1/2}$ under the area-preserving condition ($\beta=n^{-1/2}$ or $D_d=2$) is consistent with that by Winter and Tartakovsky.[23]

For the turbulent flow, the scaling exponents are different. The scaling exponents $a=2\ln n/\ln(\gamma\beta^2)$ and $b=-\ln(n\gamma)/\ln(\gamma\beta^2)$ as $n^2\beta^5/\gamma<1$, $a=-\ln(\beta^5/\gamma)/\ln(\gamma\beta^2)$ and $b=-\ln(n^3\beta^5)/\ln(\gamma\beta^2)$ as $n^2\beta^5/\gamma>1$. The same power scaling law $K_e\sim V^{1/2}$ is found under volume- and area-preserving conditions ($\gamma=n^{-1/3}$, $\beta=n^{-1/2}$). In addition, the scaling exponents can also be expressed in terms of fractal dimensions as discussed above.

Generally, the total number of branching levels $m\gg 1$ in the natural branched network (for example, the bronchial tree of mammals has about 16 branching levels). So $n=2$ and $10\leqslant m\leqslant 30$ are employed for numerical calculations in this work. The results are shown in Figs. 2–5. Figure 2 shows the scaling laws of electrical conductivity and heat conduction at different length and diameter ratios. As shown in Fig. 2(a), the fitting constants a for the electrical resistance or the thermal resistance versus the supporting volume are −0.495 at $\gamma=2^{-1/3}$ and $\beta=2^{-1/2}$, −0.165 at $\gamma=0.6$ and $\beta=0.7$, and −0.362 at $\gamma=0.6$ and $\beta=0.5$, respectively. While the scaling exponents a predicted by our model are −1/2, −0.165, and −0.365 at different microstructures. The predicted scaling exponents b between the EEC or ETC and the supporting volume are −0.096 and −0.193 at $\gamma=0.6$ and $\beta=0.5$, and $\gamma=0.7$ and $\beta=0.5$, while the fitting constants $b=-0.099$ and −0.193, respectively, see Fig. 2(b). Figure 3 indicates that the effective convective heat transfer coefficient h_e is proportional to $S^{b'}$ and V^b. The fitting constants are −0.316 and −0.211 for $\gamma=\beta=0.6$, −0.200 and −0.143 for $\gamma=0.6$ and $\beta=0.7$, and −0.206 and −0.131 for $\gamma=0.7$ and $\beta=0.6$; while the predicted scaling exponents b' are −0.322, −0.201, and −0.201, and b are −0.214, −0.142, and −0.127, respectively. Figure 4 presents the scaling laws for laminar flow at γ

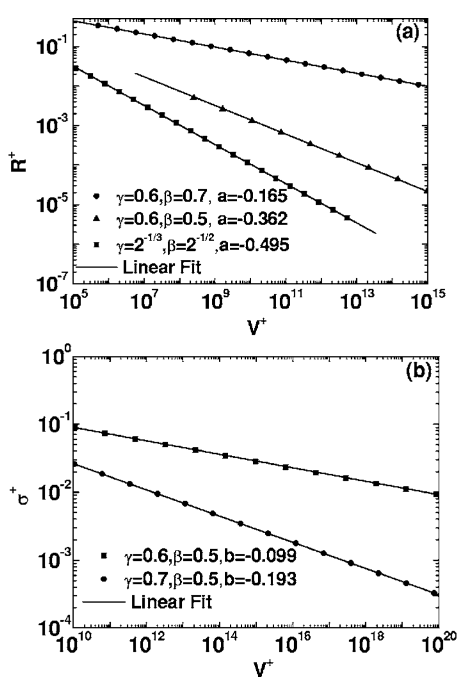

FIG. 2. The scaling laws at different length and diameter ratios: (a) The electrical resistance vs the supporting volume $R^+\sim(V^+)^a$, and (b) The electrical conductivity vs the support volume $\sigma^+\sim(V^+)^b$.

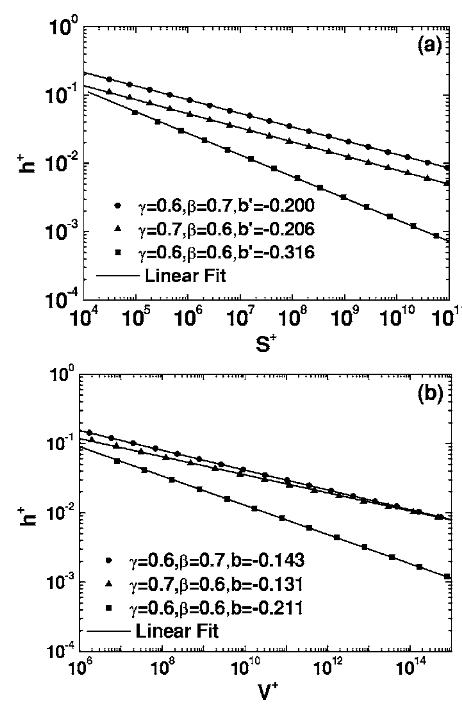

FIG. 3. The scaling laws at different length and diameter ratios: (a) The effective convective heat transfer coefficient vs the surface $h^+\sim(S^+)^{b'}$, and (b) the effective convective heat transfer coefficient vs the supporting volume $h^+\sim(V^+)^b$.

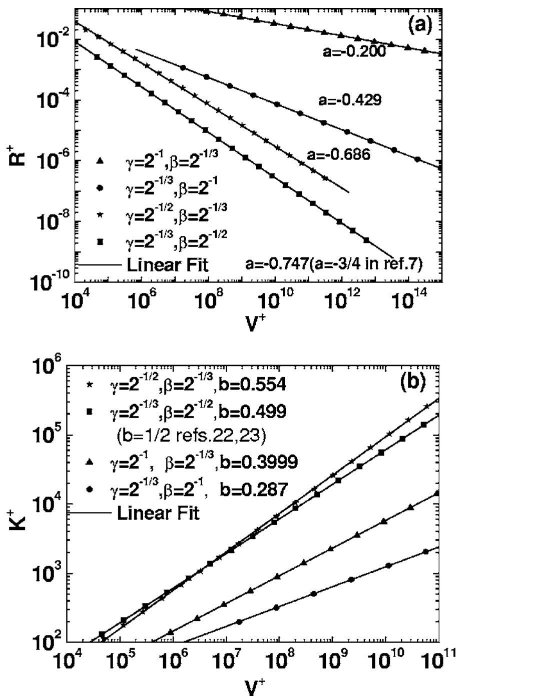

FIG. 4. The scaling laws at different length and diameter ratios: (a) The laminar flow resistance vs the supporting volume $R^+ \sim (V^+)^a$, and (b) the effective permeability vs the supporting volume $K^+ \sim (V^+)^b$.

$=2^{-1/3}$ and $\beta=2^{-1/2}$, $\gamma=2^{-1/3}$ and $\beta=2^{-1}$, $\gamma=2^{-1}$ and $\beta=2^{-1/3}$, and $\gamma=2^{-1/2}$ and $\beta=2^{-1/3}$, respectively. The fitting constants for flow resistance and volume are $a=-0.747(\approx -3/4)$, $-0.429(\approx -3/7)$, $-0.200(=-1/5)$, and $-0.686(\approx -5/7)$, and those for effective permeability and volume are $b=0.499(\approx 1/2)$, $0.287(\approx 2/7)$, $0.3999(\approx 2/5)$, and $0.554(\approx 4/7)$, respectively. While the scaling exponents

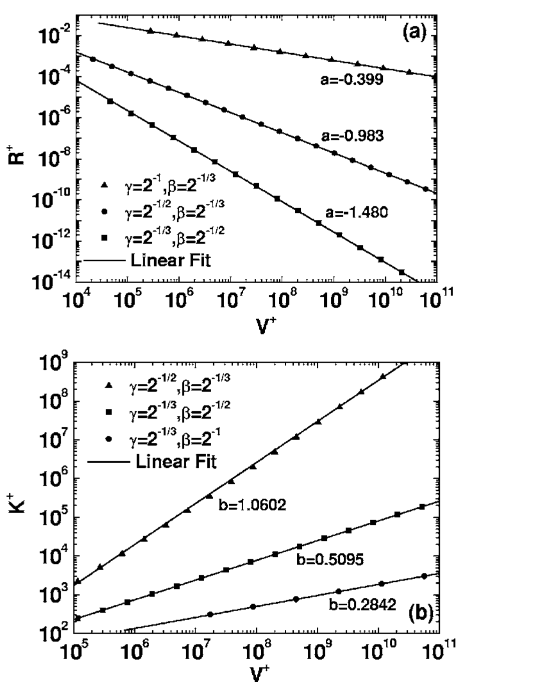

FIG. 5. The scaling laws at different length and diameter ratios: (a) The turbulent flow resistance vs the supporting volume $R^+ \sim (V^+)^a$, and (b) the effective permeability vs the supporting volume $K^+ \sim (V^+)^b$.

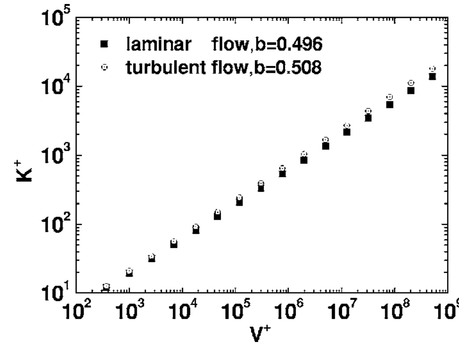

FIG. 6. The scaling law between the effective permeability and the supporting volume $K^+ \sim (V^+)^b$ at $\gamma=2^{-1/3}$ and $\beta=2^{-1/2}$ for $1 \leq m \leq 30$, the fitting constants are $b=0.496$ $(\approx 1/2)$ for laminar flow and $b=0.508$ $(\approx 1/2)$ for turbulent flow.

predicted by our model are $a=-3/4$, $-3/7$, $-1/5$, and $-5/7$; $b=1/2$, $2/7$, $2/5$, and $4/7$, respectively. Figure 5 is for the case of turbulent flow, the fitting constants for flow resistance are $a=-1.480$ $(\approx -3/2)$ at $\gamma=2^{-1/3}$ and $\beta=2^{-1/2}$, $a=-0.3999$ $(\approx -2/5)$ at $\gamma=2^{-1}$ and $\beta=2^{-1/3}$, and $a=-0.983$ (≈ -1) at $\gamma=2^{-1/2}$ and $\beta=2^{-1/3}$ [Fig. 5(a)]. While our predicted scaling exponents are $a=-3/2$, $-2/5$, and -1, respectively. Figure 5(b) indicates the scaling laws between the effective permeability for turbulent flow and the volume $(K_e \sim V^b)$ at different microstructures, and the fitting constants are $b=0.5095$ $(\approx 1/2)$ at $\gamma=2^{-1/3}$ and $\beta=2^{-1/2}$, $b=0.2842$ $(\approx 2/7)$ at $\gamma=2^{-1/3}$ and $\beta=2^{-1}$, and $b=1.0602$ $(\approx 8/7)$ at $\gamma=2^{-1/2}$ and $\beta=2^{-1/3}$, respectively. The predicted scaling exponents are $b=1/2$, $2/7$, and $8/7$, respectively. It is evident from Figs. 2–5 that the scaling exponents depend on the microstructures and are very sensitive to the microstructures (γ,β). Note that the result of $K_e \sim V^{1/2}$ under the volume- and area-preserving relations for laminar and turbulent flows is correct even for small m (see Fig. 6).

VIII. CONCLUDING REMARKS

We have derived and summarized the electrical conductivity, heat conduction, convective heat transfer, laminar flow and turbulent flow, as well as their scaling laws in the fractal-like tree networks. It is shown that the scaling laws are different for different transport properties and are very sensitive to the microstructures of the networks. The models and results we present here may be helpful for understanding the transport properties in the network structures such as the natural systems, nanotube networks, microelectronic cooling networks, organisms, fractures in oil/water reservoirs, seepage flow in porous networks/media, etc., and might provide guidance for design of composites with tree structures.

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