

Analysis of permeability for the fractal-like tree network by parallel and series models

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Abstract

This work investigates the hydraulic conductivity properties in the fractal-like tree networks between one point and a straight line. The expression for the effective permeability of the networks is derived based on the parallel and series models and the relationship between the effective permeability and the geometry structures of the network is analyzed. It is found the effective permeability after including tortuosity is about 20% lower than that without considering the tortuosity, and the tortuosity effect should be included in analysis of hydraulic conductivity properties in the networks; the effective permeability is very sensitive to the geometrical structures of the network. A comparison of the fractal-like tree network with the traditional parallel net indicates that the fractal-like tree network can provide much higher permeability than that of the traditional parallel net.

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1. Introduction

The fluid flow in tree-like branched geometry has steadily attracted many researchers in physics, biology, geological and chemical engineering, and microelectronic engineering, as well as oil recovery, etc. Examples include catalysis, flow through porous media, blood circulation, and respiration and electronic cooling. The investigations of the tree-like branched network began from 1926 since Murray proposed that the optimal ratios of the tube diameters in cardiovascular system are $2^{1/3}$, known as Murray's law [1]. Numerous subsequent researches extended Murray's work. The successive airway segments in the bronchial tree are homothetic with the approximate size ratio [2,3]. West et al. [4] discussed the origin of allometric scaling laws in biology in a rather general and detailed fashion. Bejan [5,7] Neagu and Bejan [6], and Bejan and Lorente [8,9] developed "constructal theory" by optimizing the access between one point and a finite volume, and applied the theory to the cooling of electronic devices and other engineering fields. The sequence of optimization has a definite time direction, which begins with the smallest building block to larger one with optimization at every step. For a flow system to persist in time (to survive), its configuration must evolve so that it provides easier and easier access to the currents that flow through it. It has been shown that the

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optimization leads to a tree-like network. Neagu and Bejan [6] showed that the global thermal resistance to flow between a volume and one point can be reduced to unprecedented levels by shaping the external boundary of each volume element. The volume is covered in a sequence of optimization and assembly steps that proceeds toward larger sizes. The resulting architectures are a leaf-like tree and needle-like (triangle-in-triangle) shaped structures. They also discussed the fractal-like character of these designs and their relevance to the trend toward fractal-like properties in natural-flow structures.

Most distribution systems can be described by a tree network in which the sizes of tubes regularly decrease, and in order to minimize the energy dissipated in the system, the network must be a conventionally self-similar fractal network which can be space filling [4,10]. Kearney [11] pointed out that equipment built with fractal characteristics could offer advantages over traditional fluid mixers and distributors. Chen and Cheng [12] studied the convective heat transfer and pressure drop in a fractal tree-like net of rectangular shape, and compared the network with the conventionally parallel channels. They found that the fractal tree-like network could increase the total convective heat transfer rate and reduce the total pressure drop in the fluid.

Tien and Vafai [13] presented a method for evaluation of the effective conductivity and discussed the pertinent parameters and their effects on determining the effective conductivity. In this paper, we develop the tortuosity and permeability models for flow through a fractal-like tree network between one point and a straight line presented by Lorente et al. [14] and derive an analytical expression for the effective permeability for the network based on the parallel and series models. Furthermore, we determine the relationship between the effective permeability and the geometry structures of the network, and analyze the influence of tortuosity on the effective permeability. The model and method we develop in this work might be helpful for analysis of the permeability in porous media embedded with tree network such as fractal network in oil reservoir. A comparison of the effective permeability of the network with that of the parallel channel net is also made.

2. The fundamental features of the fractal-like tree network

Tree-like networks play a unique role in science and engineering. The commonality of natural-branched networks has also been recognized in the field of fractal geometry. In fractal geometry [10], many of the geometrical features of a natural-branched network can be approximated by repeating a finite number that follows a properly designed algorithm, which results in an increasing number of channels with smaller diameters. It has been shown that mass transfer in these networks is efficient.

According to Mandelbrot [10], “trees may be called fractal in part”, and “trees are barely self-avoiding” and “these trees’ tips self-contact asymptotically”, and also according to Neagu and Bejan [6], the tree structures generated by the constructal method lead to fractals. The tree network with branching angle θ shown in Fig. 1

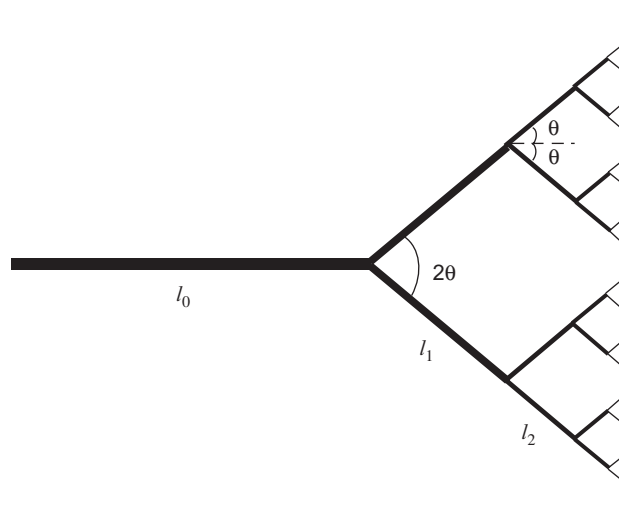


Fig. 1. Fractal-like tree network model between a point and a straight line [14].

[14] is called fractal-like tree network in this work, because the tree tips may self-contact asymptotically. The network can be obtained by repeating the postulated algorithm from the 0th level (single) channel. The network connects all the points on a straight line with a single point situated off the line. The single point may be the source of a stream (e.g. water) and the line may represent a large number of users of the water stream.

We assume that each branch of the network is a smooth cylinder, and thickness of the tube wall can be ignored. Suppose that every channel is divided into N (branching number) branches at the next level (e.g. $N = 2$ in Fig. 1) and the total number of branching levels is m . The integer $N = 2$ for dichotomy (bifurcation, pairing) is deduced from the optimization principle by Bejan [14]; therefore, we fix $N = 2$ in this study. The network presented here should be viewed as an idealized representation in which we ignore complications such as tapering of vessels and nonlinear effects. These play only a minor role in determining the properties of the entire network and could be incorporated in more detailed analysis of specific systems. A typical branch at some intermediate level k ($k = 0, 1, 2, \dots$) has length l_k and diameter d_k . To characterize the branching structures, we introduce scale factors as

$$\gamma = l_{k+1}/l_k, \quad \beta = d_{k+1}/d_k \quad (k = 0, 1, 2, \dots). \quad (1)$$

Thus, it is easy to get

$$l_k = l_0 \gamma^k, \quad d_k = d_0 \beta^k, \quad (2)$$

where l_0 and d_0 are the length and diameter of the 0th branching level, respectively. According to the fractal characteristics of the structure [10], we have

$$N = \gamma^{-D_l} = \beta^{-D_d}, \quad (3)$$

where D_l is the fractal dimension of channel length distribution, and D_d is the fractal dimension for diameter distribution (also called diameter exponent). Fractal dimension D_l varies between 1 and 2 as the scale of the density of the channels varies, while D_d is generally limited between 1 and 3, and the value $D_d = 3$ is an optimized result known as Murray's law.

3. Permeability based on parallel and series models

The model we present here provides a way to analyze the permeability of porous media which are also considered as the matrices embedded with branched channels. In this section, the effective permeability based on parallel and series models is derived. We can divide the process of derivation into two steps as shown in Fig. 2. First, all the channels of every branching level can be equivalent to a single channel using parallel model; second, all the equivalent single channels can be equivalent to a single channel by series model, then the effective permeability of the network can be calculated.

Combining Hagen–Poiseuille law with Darcy's law, Scheidegger [15] deduced the permeability of porous media by using the straight capillary model. In this work, we follow Scheidegger's method [15] and assume that the length of each channel is much larger than the diameter, and the flow through each channel is laminar

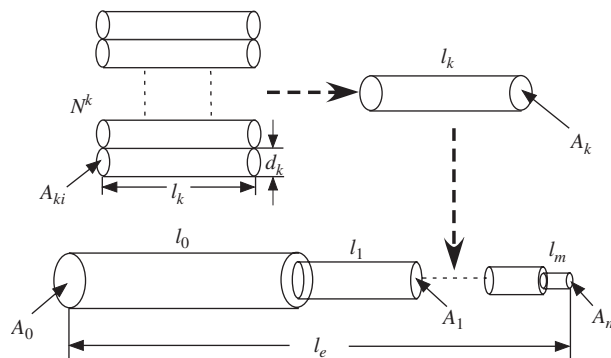


Fig. 2. Schematic of parallel and series models.

and fully developed, both thermally and hydrodynamically. According to Hagen–Poiseuille equation, the flow rate in a channel of the k th level can be written as

$$Q_k = \frac{\pi d_k^4}{128\mu} \frac{\Delta p_k}{l_k}, \tag{4}$$

where Δp_k is the pressure drop in the k th level channel, and μ is the liquid viscosity.

In porous media, Darcy’s law holds

$$Q_k = \frac{K_k}{\mu} \pi \left(\frac{d_k}{2}\right)^2 \frac{\Delta p_k}{l_k}, \tag{5}$$

where K_k is the permeability of a channel at the k th level. Comparing Eq. (4) with (5) yields

$$K_k = \frac{d_k^2}{32}, \tag{6}$$

which is exactly the expression for flow through a pipe [15].

For parallel model, the equivalent permeability K_{ek} of the k th level can be expressed as

$$K_{ek} = \frac{1}{A_{ek}} \sum_{i=0}^{N^k} A_{ki} K_{ki}, \tag{7}$$

where A_{ki} is the cross-sectional area of a channel of the k th level, A_{ek} is the equivalent cross-sectional area of the k th level, and K_{ki} is the permeability of a channel of the k th level, where

$$A_{ki} = \pi \left(\frac{d_k}{2}\right)^2, \tag{8}$$

$$A_{ek} = \sum_{i=0}^{N^k} A_{ki} = N^k \pi \left(\frac{d_k}{2}\right)^2, \tag{9}$$

$$K_{ki} = K_k = \frac{d_k^2}{32}. \tag{10}$$

Then, the equivalent permeability of the k th level can be obtained from Eq. (7) as

$$K_{ek} = \frac{d_k^2}{32}. \tag{11}$$

The network is composed of m equivalent single channels with the length $l_{ek} = l_k$, cross-sectional area A_{ek} and permeability K_{ek} in series, and then the whole network can be equivalent to a single channel with the permeability K_e . For the equivalent single channel of the network, Darcy’s law can be expressed as

$$Q = \frac{K_e}{\mu} A_e \frac{\Delta p}{l_e}, \tag{12}$$

where A_e and l_e are the equivalent cross-sectional area and length of the network, respectively.

Due to the conservation of mass, the flow rate in every level keeps unchanged. Thus, neglecting the effect of tortuosity, the pressure drop of a laminar fully developed flow in the network is

$$\Delta p = \sum_{k=0}^m \Delta p_k = \mu Q \sum_{k=0}^m \frac{l_k}{K_{ek} A_{ek}}. \tag{13}$$

Substituting Eq. (13) into Eq. (12) gives

$$K_e = l_e / \left(A_e \sum_{k=0}^m \frac{l_k}{K_{ek} A_{ek}} \right). \tag{14}$$

The equivalent length of the network is

$$l_e = \sum_{k=0}^m l_k = l_0 \frac{1 - \gamma^{m+1}}{1 - \gamma}. \quad (15)$$

According to Hagen–Poiseuille equation, the flow rate of the network can be expressed as the flow rate of the equivalent single channel as

$$Q = \frac{\pi d_e^4 \Delta p}{128 \mu l_e}, \quad (16)$$

where d_e is the equivalent diameter of the series model, thus,

$$A_e = \pi \left(\frac{d_e}{2} \right)^2. \quad (17)$$

Inserting Eq. (13) into Eq. (16) gives

$$d_e^4 = \frac{128 l_e}{\pi \sum_{k=0}^m l_k / (K_{ek} A_{ek})}. \quad (18)$$

Combining Eq. (18) and Eq. (14) with the aid of Eq. (17) yields

$$K_e = \left[\frac{l_e}{8\pi \sum_{k=0}^m l_k / (K_{ek} A_{ek})} \right]^{1/2}. \quad (19)$$

Substituting Eqs. (9), (11) and (15) into Eq. (19) with the aid of Eq. (2) gives

$$K_e = \frac{d_0^2}{32} \left[\frac{1 - \gamma^{m+1}}{1 - \gamma} \frac{1 - \gamma/N\beta^4}{1 - (\gamma/N\beta^4)^{m+1}} \right]^{1/2}. \quad (20)$$

In the above derivation, we neglect the effect of tortuosity of the network. The tortuosity is in fact an important parameter, which significantly influences the hydraulic conductivity properties. If tortuosity T is included, the dimensionless effective permeability can be expressed/defined as [17]

$$K^+ = \frac{K_e}{d_0^2/32} \frac{1}{T}. \quad (21)$$

The tortuosity T is often defined by [16,17]

$$T = \frac{L_t}{L_0}, \quad (22)$$

where L_t and L_0 are the actual length of flow path and the straight length or thickness of a sample along the macroscopic pressure gradient, respectively. From Fig. 1, L_t and L_0 can be expressed as

$$L_t = \sum_{k=0}^m l_k = l_0 \frac{1 - \gamma^{m+1}}{1 - \gamma}, \quad (23)$$

$$L_0 = l_0 + \sum_{k=1}^m l_k \cos \theta = l_0 \left[1 + \frac{\gamma(1 - \gamma^m)}{1 - \gamma} \cos \theta \right]. \quad (24)$$

So, inserting Eqs. (23) and (24) into Eq. (22), the tortuosity can be written as

$$T = \frac{1 - \gamma^{m+1}}{(1 - \gamma)[1 + \gamma(1 - \gamma^m)/(1 - \gamma) \cos \theta]}. \quad (25)$$

Substituting Eqs. (20) and (25) into Eq. (21) results in

$$K^+ = \left[\frac{1 - \gamma}{1 - \gamma^{m+1}} \frac{1 - \gamma/N\beta^4}{1 - (\gamma/N\beta^4)^{m+1}} \right]^{1/2} \left[1 + \frac{\gamma(1 - \gamma^m)}{1 - \gamma} \cos \theta \right]. \tag{26}$$

It can be seen from Eq. (26) that the value of the dimensionless effective permeability of the network K^+ is 1 as $m = 0$ and $\gamma = 0$, i.e. the effective permeability of the network is reduced to the permeability of a single channel. And as $\beta \gg 1$ the term $[1 - \gamma/N\beta^4]/[1 - (\gamma/N\beta^4)^{m+1}]$ in Eq. (26) tends to be 1, and then the effective permeability of the network is independent of the diameter ratio β . These indicate that our model is consistent with the physical situation. The relationship between the effective permeability and fractal dimensions can be obtained by substituting Eq. (3) into Eq. (26),

$$K^+ = \left[\frac{1 - N^{-1/D_l}}{1 - N^{-(m+1)/D_l}} \frac{1 - N^{(4/D_d - 1/D_l - 1)}}{1 - N^{(4/D_d - 1/D_l - 1)(m+1)}} \right]^{1/2} \left[1 + \frac{N^{-1/D_l}(1 - N^{-m/D_l})}{1 - N^{-1/D_l}} \cos \theta \right]. \tag{27}$$

Eqs. (26) and (27) reveal that the dimensionless effective permeability of the networks depends on the geometrical structures and the fractal dimensions of the networks.

In order to compare the permeability with the parallel channels, the flow rates are assumed to be the same and the velocity of the parallel channel net is also assumed to be the same v_0 . The number of parallel channels with total branching levels m is N^m (i.e. the number of parallel channels is equal to the total number of outlets in the fractal-like tree network), and l and d are the length and diameter of the parallel channel net, respectively. Thus,

$$v_0 \frac{\pi}{4} d_0^2 = N^m v_0 \frac{\pi}{4} d^2 \tag{28}$$

and consequently,

$$d = N^{-m/2} d_0. \tag{29}$$

For the parallel channel net composed of N^m channels, the equivalent permeability based on parallel model can be obtained

$$K_{ep} = \frac{d^2}{32 T_p} \frac{1}{N^m}, \tag{30}$$

where T_p is the tortuosity of the parallel channel net which is the same as the network. Then, K_{ep} can be expressed as

$$K_{ep} = \frac{d_0^2}{32 N^m} \frac{1}{1 - \gamma^{m+1}} \left[1 + \frac{\gamma(1 - \gamma^m)}{1 - \gamma} \cos \theta \right]. \tag{31}$$

According to Eq. (26), the equivalent permeability after considering totuosity of the fractal-like tree network can be written as

$$K_{eb} = \frac{d_0^2}{32} \left[\frac{1 - \gamma}{1 - \gamma^{m+1}} \frac{1 - \gamma/N\beta^4}{1 - (\gamma/N\beta^4)^{m+1}} \right]^{1/2} \left[1 + \frac{\gamma(1 - \gamma^m)}{1 - \gamma} \cos \theta \right]. \tag{32}$$

Combining Eqs. (31) with (32) results in

$$K^{+'} = \frac{K_{eb}}{K_{ep}} = N^m \left[\frac{1 - \gamma^{m+1}}{1 - \gamma} \frac{1 - \gamma/N\beta^4}{1 - (\gamma/N\beta^4)^{m+1}} \right]^{1/2}, \tag{33}$$

which is the expression for dimensionless permeability.

4. Results and discussions

In this section, we compute the tortuosity, dimensionless effective permeability and discuss the effect of the geometrical structures of the network on the effective permeability. Since the limits of the fractal dimensions mentioned above, the scale factors γ and β are limited accordingly. Fig. 3 displays the tortuosity versus the length ratios γ , total number of branching levels m and branching angles θ . It is evident that the values of tortuosity are always greater than 1.0. Fig. 3(a) shows that the tortuosity of the network increases with the increase of length ratio γ when m and θ are given; the higher the branching levels, the larger the tortuosity. Fig. 3(b) reveals that the tortuosities T reach the different asymptotical values as the branching levels m arrive at certain values, depending on the length ratios γ , e.g. as $\gamma = 0.55, m = 6; \gamma = 0.70, m = 12$. But, at low branching levels, the tortuosity is very sensitive to the branching levels m and increases drastically with m . The mechanism behind this phenomenon is not well understood. Fig. 3(c) indicates that the tortuosity increases

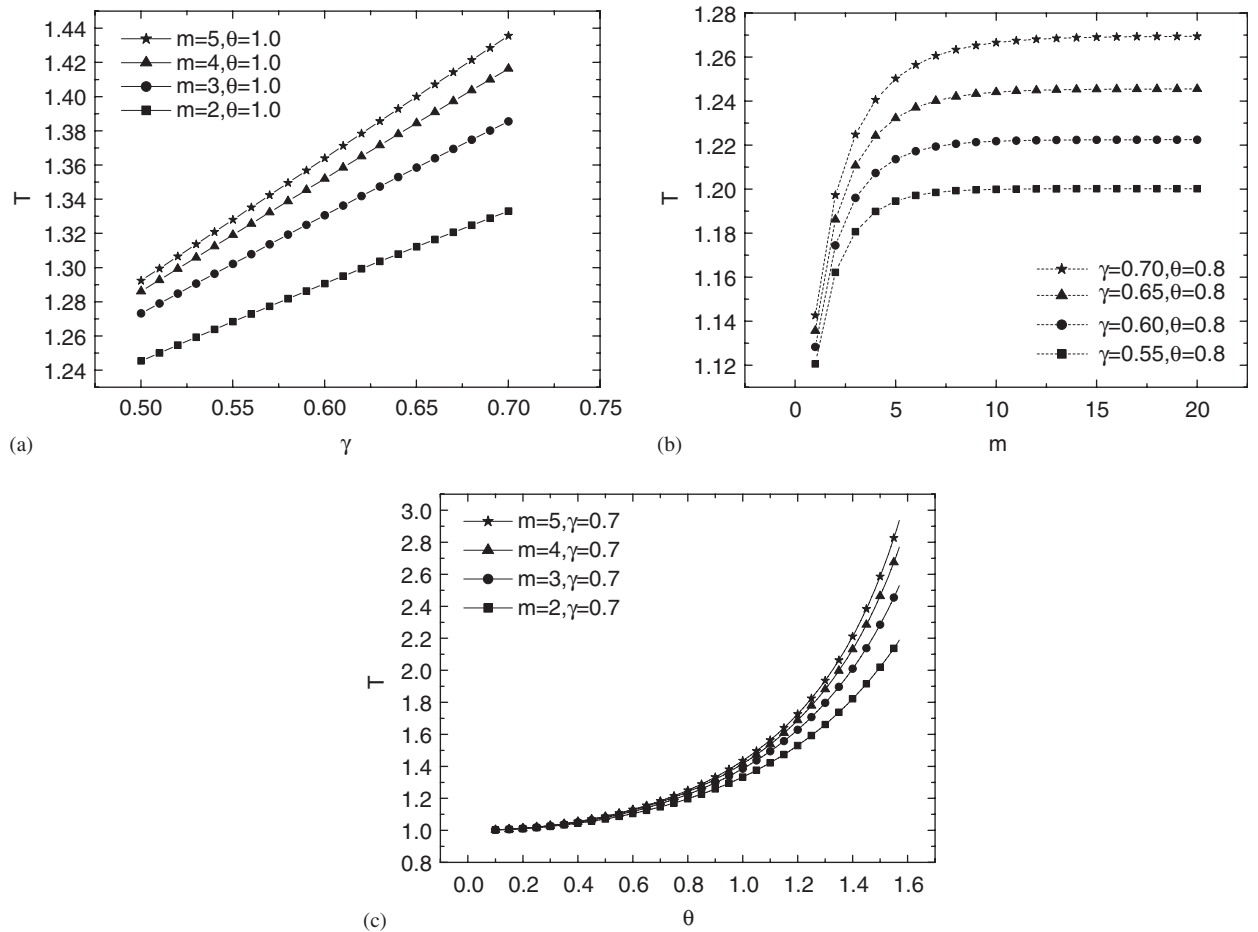


Fig. 3. Analysis of the tortuosity of the fractal-like tree network. Plot of T versus (a) γ at $\theta = 1.0$ and $m = 2, 3, 4, 5$, (b) m at $\theta = 0.8$ and $\gamma = 0.55, 0.60, 0.65, 0.70$, and (c) θ at $\gamma = 0.7$ and $m = 2, 3, 4, 5$.

with the branching angle θ , the larger the branching angle θ , the larger the tortuosity. It is also seen that the tortuosity tends to be 1.0 at smaller branching angle, and this is consistent with practical situation. In all, Fig. 3 reveals that the geometry parameters, γ , θ and m , have a significant influence on the tortuosity, which thus have a significant effect on flow resistance/permeability for flow through fractal-like tree branched networks. In other words, the effect of the tortuosity on resistance/permeability should not be neglected in fractal-like tree networks.

Fig. 4 denotes the variation of dimensionless effective permeability with the geometrical parameters without considering the effect of tortuosity of the network. From Figs. 4(a) and (b), it can be seen that the effective permeability of the network decreases with the increase of length ratio γ and increases with the increase of diameter ratio β . These are expected because the larger length ratio γ means the longer daughter branches, leading to the higher resistance and lower permeability; while, the larger diameter ratio β means the wider daughter branches, leading to the lower resistance and higher permeability. Fig. 4(c) shows that the effective permeability decreases with the increase of branching levels m , and this is also expected because a network with higher branching levels m means denser branches of the network, causing the higher resistance and lower permeability. However, as the branching levels continuously increase, the permeability approaches an asymptotical value. Eq. (20) indicates that the permeability is independent of the branching angle θ if the tortuosity is not included in the permeability. However, in reality, the flow resistance is related to the

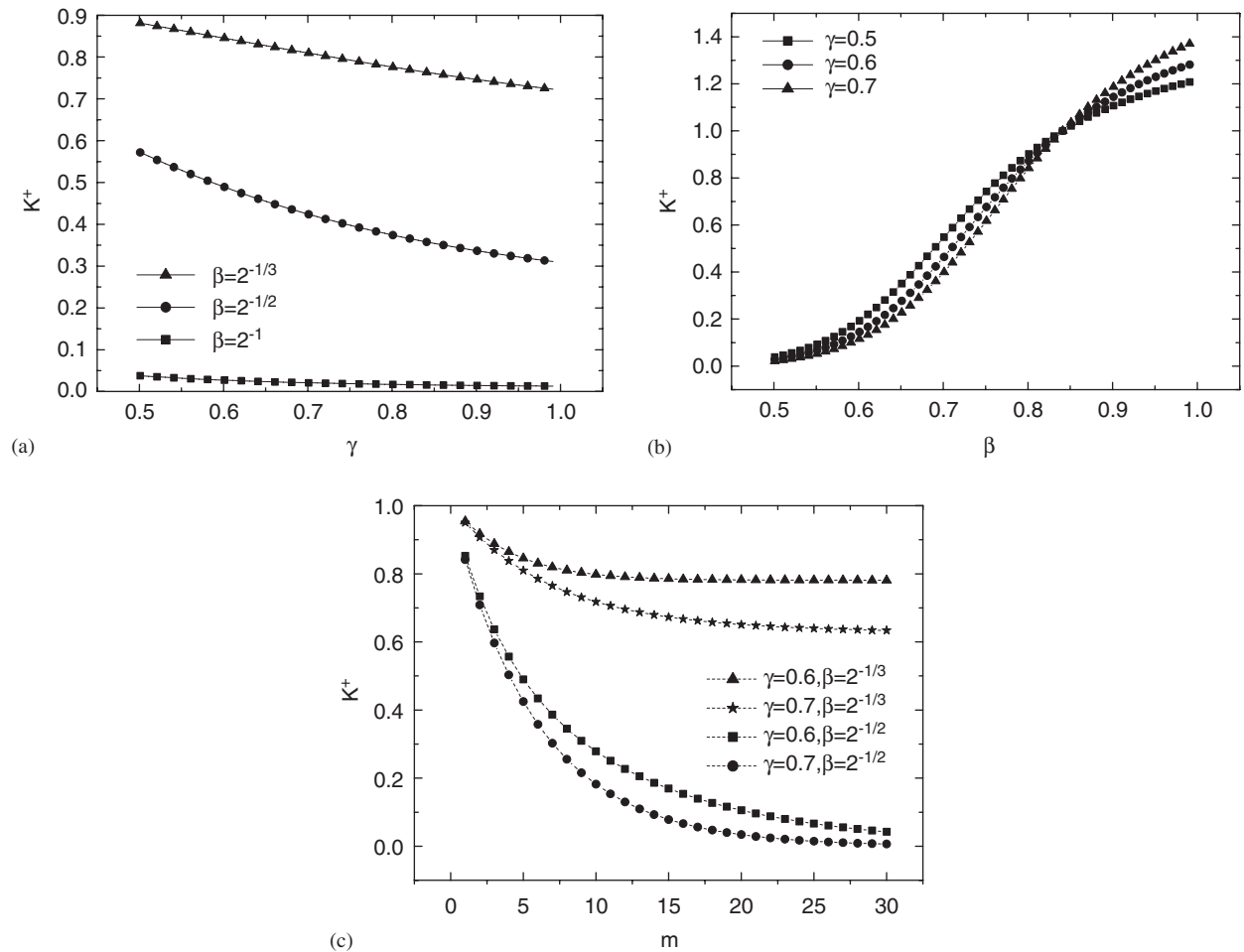


Fig. 4. The dimensionless effective permeability K^+ neglecting tortuosity of the network versus (a) γ for different β at $n = 2, m = 5$, (b) β for different γ at $n = 2, m = 5$, and (c) m for different γ and β .

branching angle and thus permeability is dependent upon the branching angle. Therefore, the effect of tortuosity on permeability should not be neglected.

If the tortuosity effect is taken into account, the relations among the dimensionless effective permeability, length and diameter ratio, the number of branching levels, tortuosity and branching angle are shown in Fig. 5. It is found from Fig. 5 that the dimensionless effective permeability K^+ of the network is always less than 1.0, and small variations in geometrical structures in the network can induce very large variations in the effective permeability, which is much similar to the net air flux in bronchial tree [3]. Compared with Fig. 4, the similar variation tendency of the effective permeability is found in Fig. 5. But the effective permeability, after taking into account the tortuosity effect, is about 20% lower than that without considering the tortuosity effect, see Figs. 5(a) and (b). Figs. 5(c) and (d) clearly show that the branching angle θ has a significant influence on the effective permeability, and the effective permeability decreases with the increase of θ . This is consistent with the physical situation, and the tortuosity is thus an important geometrical parameter which should be included in a proper model for hydraulic conductivity properties in the networks.

The comparisons, based on Eq. (33), between the present model and the parallel channel model for permeability are presented in Fig. 6. The results are qualitatively consistent with those by Chen and Cheng [12]. Fig. 6 shows that the effective permeability of the fractal-like tree network is much larger than that of the parallel net under the condition of the same flow rate. The larger the length ratio γ , the lower the dimensionless permeability. The increase of the diameter ratio β creates a remarkable increase in the

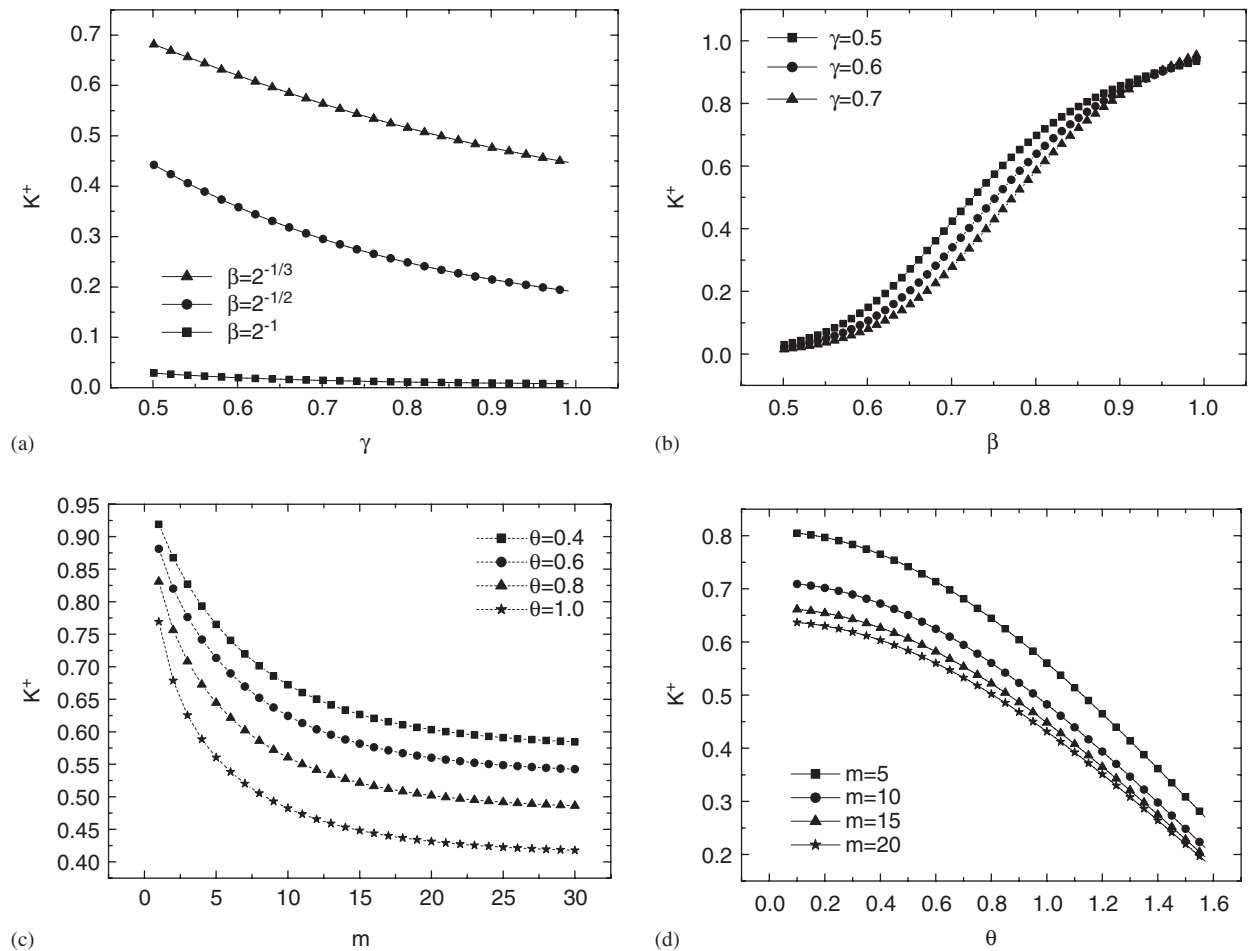


Fig. 5. The dimensionless effective permeability K^+ in view of tortuosity of the network versus (a) γ for different β at $n = 2, m = 5$, (b) β for different γ at $n = 2, m = 5$, (c) m for different θ at $\gamma = 2^{-1/2}, \beta = 2^{-1/3}$, (d) θ for different m at $\gamma = 2^{-1/2}, \beta = 2^{-1/3}$.

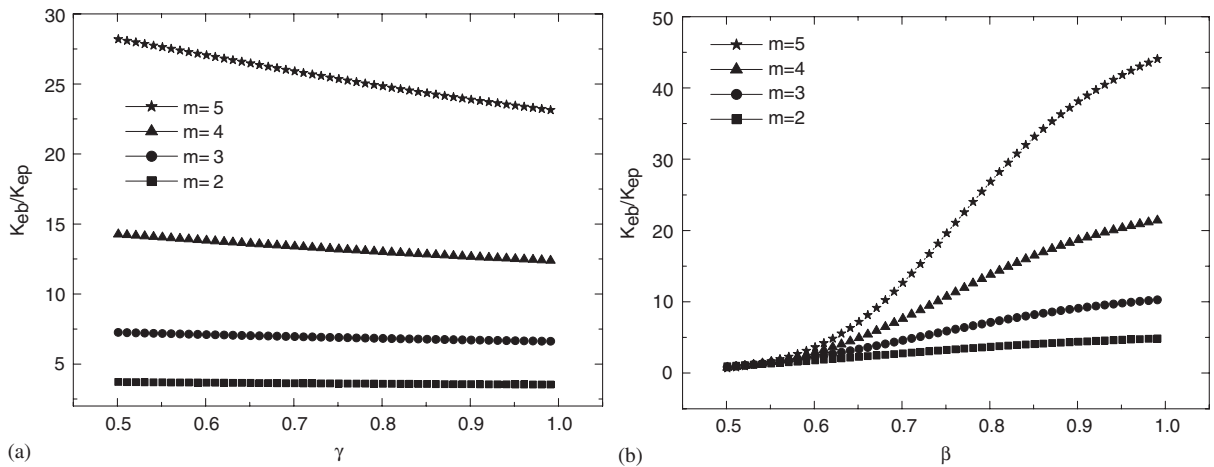


Fig. 6. Schematic variation of the dimensionless permeability K_{eb}/K_{ep} with (a) the length ratio γ , and (b) the diameter ratio β at $m = 2, 3, 4, 5$.

dimensionless permeability, e.g. a 10% increase in the diameter ratio β would nearly double the dimensionless permeability. Moreover, the larger the branching levels m , the larger the dimensionless permeability. It is evident that the fractal-like tree networks can significantly increase the permeability compared to the parallel channel net.

5. Concluding remarks

In this paper, the flow through the fractal-like tree networks between a single point and a straight line is investigated and the effective permeability of the networks is derived. A comparison of effective permeability is made between the fractal-like tree networks with the traditional parallel channel net. The tortuosity should be included in analysis of the permeability of the networks. The fractal-like tree networks can provide much higher permeability than that of the traditional parallel channel net and the advantage is more obvious with the higher branching levels m . It is found that the permeability decreases with the increase of length ratio γ , total number of branching levels m , as well as the branching angle θ . The permeability increases with the increase of diameter ratio β , and it approximately reaches an asymptotical value when total number of branching levels m is greater than a certain value, depending on the branching length and diameter ratios. The effective permeability after including tortuosity is about 20% lower than that without considering the tortuosity, and the tortuosity effect should be included in analysis of hydraulic conductivity properties in the networks. It is also found that small variations in geometrical structures in the network can induce very large variations in the effective permeability.

Acknowledgments

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